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Mass Matrix Model Broken From A_4 To $2 \leftrightarrow 3$ Symmetry

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Abstract

$2 \leftrightarrow 3$ symmetry is realized by the breaking from alternating group of degree 4 (A_4) symmetry. A_4 explains why the generation number is three. However the mass matrices are realized in the form of the breaking to $2 \leftrightarrow 3$ symmetry $\times Z_3$, which leads us to $2 \leftrightarrow 3$ symmetric mass matrix with vanishing (1,1) component. Thus the 3×3 mass matrix model with $2 \leftrightarrow 3$ symmetry and vanishing (1,1) component has the group theoretical background as the symmetry in GUT model.

Keywords : Family symmetry, mass matrix model,

1 Introduction

There have been many discussions on GUTs in these decades. These study the inter-relations between quarks and leptons mass matrices. For instance, the renormalizable minimal $SO(10)$ GUT predicts the mass relations [1] [2]

$$\begin{aligned} M_u &= c_{10}M_{10} + c_{126}M_{126} \\ M_d &= M_{10} + M_{126} \\ M_D &= c_{10}M_{10} - 3c_{126}M_{126} \\ M_e &= M_{10} - 3M_{126} \\ M_L &= c_L M_{126} \\ M_R &= c_R M_{126} , \end{aligned} \tag{1.1}$$

Here M_u , M_d , M_D , M_e , M_L , M_R are 3×3 up type quark, down type quark, Dirac neutrino, charged lepton, left-handed Majorana, and right-handed Majora neutrino mass matrices, respectively. They are composed of only two kinds of mass matrices M_{10} and M_{126} . Thus SO(10) group properties predict the combination of M_{10} and M_{126} between quark lepton mass matrices but do not predict any structure of M_{10} and M_{126} themselves. Interrelations between different families (so called family symmetry) become the recent topics. One of the most important problems of family symmetries may be why we have three families. This may be partly solved by A_4 group since it leads us to triplet Higgs [3]. A_4 is the four degree symmetry group with even permutation whose elements we denote as (a_1, a_2, a_3, a_4) . A_4 is generated by the S and T and their products, which satisfy

$$S^2 = T^3 = (ST)^3 = 1 \quad (1.2)$$

The three-dimensional unitary representation, in a basis where the element S is diagonal, is built up from:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1.3)$$

Let us practice the transformation, for instance, ST to $V \equiv (a_1, a_2, a_3)^T$

$$TV = \begin{pmatrix} a_2 \\ a_3 \\ a_1 \end{pmatrix} \quad (1.4)$$

$$STV = \begin{pmatrix} a_2 \\ -a_3 \\ -a_1 \end{pmatrix} \quad (1.5)$$

The rule of the game for reading the permutation group of four degree from three dimensional vector is to make plus element change to a_4 and do minus signs interchange, and ST corresponds to (a_4, a_1, a_3, a_2) . Thus S means the $2 \leftrightarrow 3$ symmetry and T does cyclic permutation or equivalently Z_3 .

Mathematically this is the elementary example of Sylow's theorem [4]: The order of A_4 is $12 = 2^2 \times 3$, and it is the product of normal subgroup V_4 , composed of

$$(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), 1, \quad (1.6)$$

and Z_3 . Thus $A_4 \supset (2 \leftrightarrow 3)\text{symmetry} \times Z_3$. This fact is very important for the model buiding as will be shown soon.

Unfortunately A_4 is too restrictive for the universal structures of quark and lepton mass matrices. $2 \leftrightarrow 3$ symmetry was first proposed by us as the neutrino mass matrix model [5] and soon be extended to quark sectors by incorporating CP phases [6].

1.1 Z_3 symmetry

Let us start from Z_3 symmetry. We assign Z_3 charge of each generation of fermions as

$$\begin{aligned}\psi_{1L} &\rightarrow \psi_{1L}, \\ \psi_{2L} &\rightarrow \omega \psi_{2L}, \\ \psi_{3L} &\rightarrow \omega^2 \psi_{3L},\end{aligned}\tag{1.7}$$

where $\omega^3 = +1$. Then, the bilinear terms $\bar{q}_{Li} u_{Rj}$, $\bar{q}_{Li} d_{Rj}$, $\bar{\ell}_{Li} \nu_{Rj}$, $\bar{\ell}_{Li} e_{Rj}$ and $\bar{\nu}_{Ri}^c \nu_{Rj}$ [$\nu_R^c = (\nu_R)^c = C \bar{\nu}_R^T$ and $\bar{\nu}_R^c = \overline{(\nu_R^c)}$] are transformed as follows:

$$\begin{pmatrix} 1 & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega \\ \omega^2 & \omega & \omega \end{pmatrix},\tag{1.8}$$

where

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \ell_L = \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}.\tag{1.9}$$

Therefore, if we assume two SU(2) doublet Higgs scalars H_1 and H_2 (as we will soon argue, this is the case of realistic models), which are transformed as

$$H_1 \rightarrow \omega H_1, \quad H_2 \rightarrow \omega^2 H_2,\tag{1.10}$$

the Yukawa interactions are given as follows

$$\begin{aligned}H_{\text{int}} &= \sum_{A=1,2} \left(Y_{(A)ij}^u \bar{q}_{Li} \widetilde{H}_A u_{Rj} + Y_{(A)ij}^d \bar{q}_{Li} H_A d_{Rj} \right) \\ &+ \sum_{A=1,2} \left(Y_{(A)ij}^\nu \bar{\ell}_{Li} \widetilde{H}_A \nu_{Rj} + Y_{(A)ij}^e \bar{\ell}_{Li} H_A e_{Rj} \right) \\ &+ \left(Y_{(1)ij}^R \bar{\nu}_{Ri}^c \widetilde{\Phi}^0 \nu_{Rj} + Y_{(2)ij}^R \bar{\nu}_{Ri}^c \Phi^0 \nu_{Rj} \right) + \text{h.c.},\end{aligned}\tag{1.11}$$

where

$$H_A = \begin{pmatrix} H_A^+ \\ H_A^0 \end{pmatrix}, \quad \widetilde{H}_A = \begin{pmatrix} \overline{H}_A^0 \\ -H_A^- \end{pmatrix},\tag{1.12}$$

Therefore,

$$Y_{(1)}^u, Y_{(2)}^d, Y_{(1)}^\nu, Y_{(2)}^e, Y_{(2)}^R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad Y_{(2)}^u, Y_{(1)}^d, Y_{(2)}^\nu, Y_{(1)}^e, Y_{(1)}^R = \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.\tag{1.13}$$

In (1.13), the symbol $*$ denotes non-zero quantities. Here, in order to give heavy Majorana masses of the right-handed neutrinos ν_R , we have assumed an SU(2) singlet Higgs scalar Φ^0 , which is transformed as H_1 . Mass matrices are sum of $Y_{(1)}$ and $Y_{(2)}$ and their (1,1) element must be vanished:

$$\begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.\tag{1.14}$$

For the minimal $SO(10)$ GUT two sets of doublets come from $(\mathbf{1}, \mathbf{2}, \mathbf{2}) \subset \mathbf{10}$ and $(\overline{\mathbf{15}}, \mathbf{2}, \mathbf{2}) \subset \overline{\mathbf{126}}$ under $SU(4)_c \times SU(2)_L \times SU(2)_R$, and Yukawa coupling is

$$\begin{aligned} W_Y &= \overline{u}_i \left(Y_{10}^{ij} H_{10}^u + Y_{126}^{ij} H_{126}^u \right) q_j + \overline{d}_i \left(Y_{10}^{ij} H_{10}^d + Y_{126}^{ij} H_{126}^d \right) q_j \\ &+ \overline{\nu}_i \left(Y_{10}^{ij} H_{10}^u - 3Y_{126}^{ij} H_{126}^u \right) \ell_j + \overline{e}_i \left(Y_{10}^{ij} H_{10}^d - 3Y_{126}^{ij} H_{126}^d \right) \ell_j \\ &+ \overline{\nu}_i \left(Y_{126}^{ij} v_R \right) \overline{\nu}_j, \end{aligned} \quad (1.15)$$

Thus in this case we have not two Higgs doublets but two sets of Higgs doublets,

$(H_1 = H_{10}^u, H_2 = H_{126}^u)$ and $(\widetilde{H}_1 = H_{10}^d, \widetilde{H}_2 = H_{126}^d)$. However the transformation properties are same and the above discussions remains valid. v_R in (1.15) is the vev of $(\mathbf{15}, \mathbf{1}, \mathbf{3}) \subset \mathbf{126}$, a singlet under Standard gauge group.

The remaining $2 \leftrightarrow 3$ symmetry gives the constraint on $*$ in (1.14).

1.2 $2 \leftrightarrow 3$ symmetry

$2 \leftrightarrow 3$ symmetry in addition to Z_3 gives the constraint to mass matrix [6] \widehat{M}_f as

$$\widehat{M}_f = \begin{pmatrix} 0 & A_f & A_f \\ A'_f & B_f & C_f \\ A'_f & C_f & B_f \end{pmatrix}, \quad (1.16)$$

where A_f, A'_f, B_f , and C_f are real parameters.

This matrix is block diagonalized by the maximal mixing O' between 2'nd and 3'rd generation

$$O' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \quad (1.17)$$

as

$$O'^T \widehat{M}_f O' = \begin{pmatrix} 0 & \sqrt{2}A & 0 \\ \sqrt{2}A & B+C & 0 \\ 0 & 0 & B-C \end{pmatrix} \quad (1.18)$$

This is diagonalized by two orthogonal matrices O_{f1} and O_{f2} as

$$O_{f1}^T \widehat{M}_f O_{f2} = \text{diag}(m_{f1}, m_{f2}, m_{f3}), \quad (1.19)$$

where m_{f1}, m_{f2} , and m_{f3} are eigenvalues of M_f . The orthogonal matrices O_{f1} and O_{f2} are given by

$$O_{f1} = O' \begin{pmatrix} \cos\varphi_{f1} & -\sin\varphi_{f1} & 0 \\ \sin\varphi_{f1} & \cos\varphi_{f1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1.20)$$

$$O_{f2} = O' \begin{pmatrix} \cos\varphi_{f2} & -\sin\varphi_{f2} & 0 \\ \sin\varphi_{f2} & \cos\varphi_{f2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.21)$$

If $\varphi_{f1} = \varphi_{f2} = -\pi/6$, mixing gives tribimaximal.

$$O = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (1.22)$$

That is, the maximal subgroup $2 \leftrightarrow 3$ symmetry is the symmetric matrix $A'_f = A_f$, and therefore $O_{f1} = O_{f2}$. That is, $2 \leftrightarrow 3$ symmetry proposed in [5] is formulated in the subgroup of $A4$ model. As is well known, $A4$ symmetry is considered as the model of lepton sector and has not found the successful generalization to quark sector. If we incorporate the CP phases which is out of $A4$ arguments, we can give the consistent predictions of quark mass matrices as well as of lepton sectors [6]

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